Computer Graphics and Programming

Quaternion

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2020/10/22

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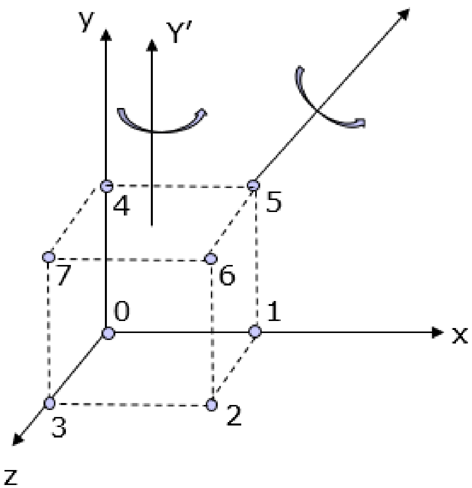
Problem of Rotation along an Arbitrary Axis



Rotation Along 3-5

Y'=RotX(Y)

RotZ(Y')

Complex Angle Calculation

- Homogeneous Transform needs
  - We find proper Rotation and Rotation for an Arbitrary Axis → It is NOT Easy
  - Reminds HW 10
- How we do it Easily → Quaternion by Hamilton

Quaternion

$$q = \underbrace{s}_{\text{real}} + \underbrace{xi + yj + zk}_{\text{imaginary}} \qquad q \in Z$$

- Quaternion is a Four Dimensional Complex Number
  - Remind we learn 2 Dim Complex number

$$z = x + yi \qquad z \in Z$$

- Quaternion has
  - 1 scalar value(Real Part) $\mathrm{Re}(q) = s$
  - 3 dimensional Imaginary Part
  - Imaginary pat is a 3 Dim. vector $\mathrm{Im}(q) = (x, y, z)$

Quaternion has three imaginary part, *i, j, k*

$$q = s + xi + yj + zk$$

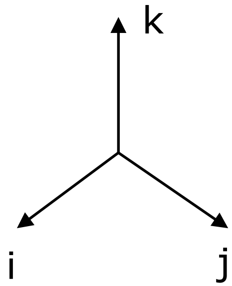
$$i \cdot i = -1$$

$$j \cdot j = -1$$

$$k \cdot k = -1$$

Same as in Complex number

3D Axis



$$i \cdot j = k$$

$$j \cdot k = i$$

$$k \cdot i = j$$

Defining i*j = k that i, j, and k are orthogonal

- Main idea is that i, j, and k are Orthogonal as in XYZ 3D space

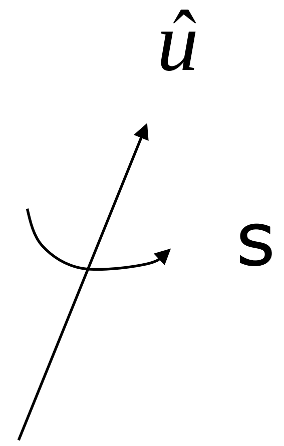
$$i \perp j, \;\; j \perp k, \;\; k \perp i$$

- Similar to Cross Product.

$$j \cdot i = -k, \;\; k \cdot j = -i, \;\; i \cdot k = -j$$

Quaternion Basic Form

$$q = s + xi + yj + zk$$
$$= s + (xi + yj + zk)$$
$$= (s, \hat{u})$$



- Quaternion has one scalar(Angle) and one Vector(axis)
- Quaternion Addition

$$q_1 = s_1 + x_1 i + y_1 j + z_1 k, \quad q_2 = s_2 + x_2 i + y_2 j + z_2 k$$
$$\therefore q_1 + q_2 = (s_1 + s_2) + (x_1 + x_2)i + (y_1 + y_2)j + (z_1 + z_2)k$$

Quaternion Multiplication (1)

$$q_1 = s_1 + x_1 i + y_1 j + z_1 k, \quad q_2 = s_2 + x_2 i + y_2 j + z_2 k$$

$$\begin{aligned}
q_1 \cdot q_2 &= (s_1 + x_1 i + y_1 j + z_1 k) \cdot (s_2 + x_2 i + y_2 j + z_2 k) \\
&= s_1 s_2 + s_1 x_2 i + s_1 y_2 j + s_1 z_2 k + \\
&\quad x_1 s_2 i + x_1 x_2 i^2 + x_1 y_2 ij + x_1 z_2 ik + \\
&\quad y_1 s_2 j + y_1 x_2 ji + y_1 y_2 j^2 + y_1 z_2 jk + \\
&\quad z_1 s_2 k + z_1 x_2 ki + z_1 y_2 kj + z_1 z_2 k^2
\end{aligned}$$

$$i \cdot j = k \qquad j \cdot i = -k$$

$$j \cdot k = i \qquad k \cdot j = -i$$

$$k \cdot i = j \qquad i \cdot k = -j$$

$$\begin{aligned}
&= s_1 s_2 + s_1 x_2 i + s_1 y_2 j + s_1 z_2 k + \\
&\quad x_1 s_2 i - x_1 x_2 + x_1 y_2 k - x_1 z_2 j + \\
&\quad y_1 s_2 j - y_1 x_2 k - y_1 y_2 + y_1 z_2 i + \\
&\quad z_1 s_2 k + z_1 x_2 j - z_1 y_2 i - z_1 z_2
\end{aligned}$$

Quaternion Multiplication (2)

$$q_1 \cdot q_2 = \left[s_1 s_2 - x_1 x_2 - y_1 y_2 - z_1 z_2 \right] +$$
$$s_1 x_2 i + s_1 y_2 j + s_1 z_2 k + s_2 x_1 i + s_2 y_1 j + s_2 z_1 k +$$
$$\left(y_1 z_2 - z_1 y_2 \right) i + \left(z_1 x_2 - x_1 z_2 \right) j + \left(x_1 y_2 - y_1 x_2 \right) k$$

$$q_1 \cdot q_2 = \left[s_1 s_2 - \hat{u}_1 \bullet \hat{u}_2 \right] + \left[s_1 \hat{u}_2 + s_2 \hat{u}_1 + \hat{u}_1 \times \hat{u}_2 \right] = s' + \hat{u}' \in \mathrm{Q}$$

- Magnitude of Quaternion
  - Remind |Z| $\quad | z |^2 = | z\bar{z} |$

$$\bar{q} = (s, -\hat{u})$$
$$q \cdot \bar{q} = \left[ss - \hat{u} \bullet (-\hat{u}) \right] + \left[s\hat{u} - s\hat{u} + \hat{u} \times -\hat{u} \right]$$
$$= s^2 + \hat{u} \bullet \hat{u}$$

Unit Quaternion

- Magnitude of Complex Variable Quaternion |q|
  - Remind |z| $|z|^2 = |z\bar{z}|$
  - Magnitude of Quaternion $q = (s, \hat{u}) \quad \bar{q} = (s, -\hat{u})$

$$q_1 \cdot q_2 = \left[s_1 s_2 - \hat{u}_1 \bullet \hat{u}_2\right] + \left[s_1 \hat{u}_2 + s_2 \hat{u}_1 + \hat{u}_1 \times \hat{u}_2\right] = s' + \hat{u}' \in Q$$

$$q \cdot \bar{q} = \left[ss - \hat{u} \bullet (-\hat{u})\right] + \left[s\hat{u} - s\hat{u} + \hat{u} \times -\hat{u}\right]$$

$$= s^2 + \hat{u} \bullet \hat{u}$$

- Definition of Pure Quaternion: s=0

$$q_p = (0, \hat{u}) \qquad \therefore \ |q \cdot \bar{q}| = |\hat{u} \bullet \hat{u}|$$

- Definition of Unit Quaternion(Versor) : $|q_u| = 1$

$$|q \cdot \bar{q}| = |s^2 + \hat{u} \bullet \hat{u}| = 1$$

Inverse Quaternion

- Inverse Quaternion is derived from Magnitude Equation

$$\|q\|^2 = q \cdot \bar{q}$$

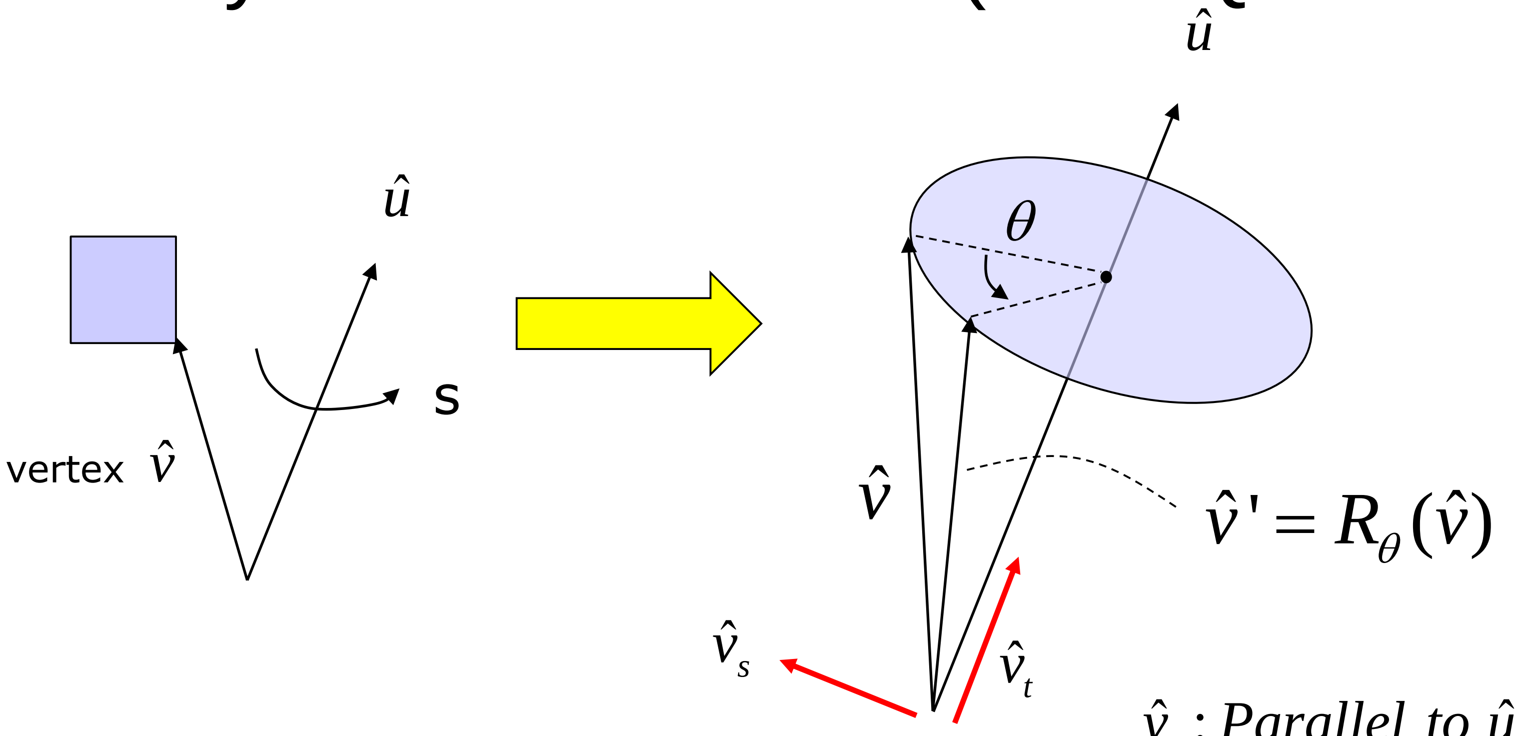
$$\therefore q^{-1} = \frac{\bar{q}}{\|q\|^2}$$

- If q is an unit quaternion,

$$\|q\|^2 = q \cdot \bar{q}$$

$$\therefore q^{-1} = \frac{\bar{q}}{\|q\|^2} = \bar{q}$$

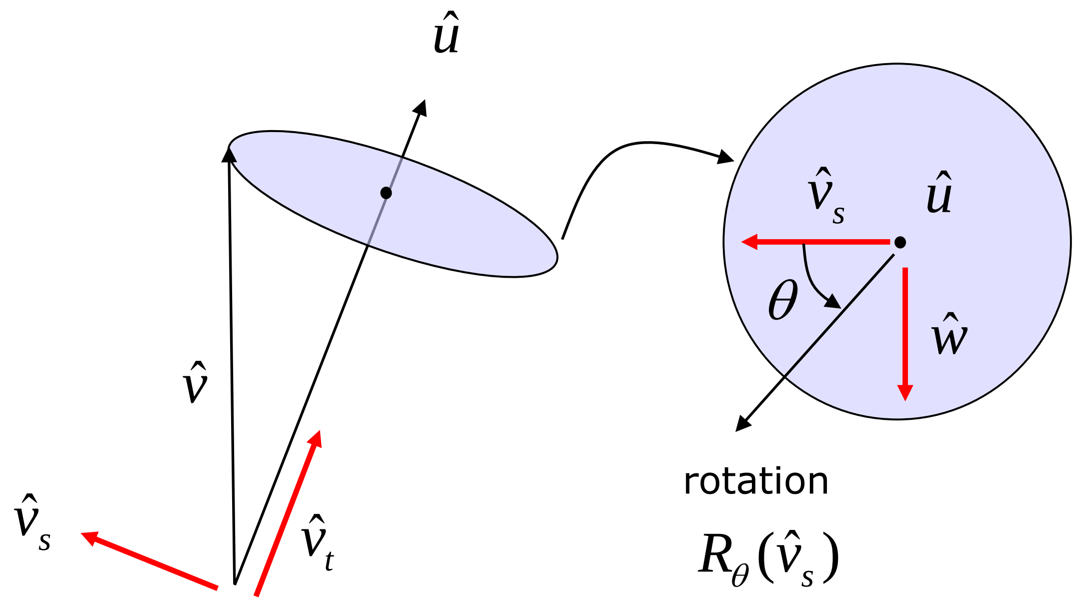
Let's go Back to Arbitrary Rotation by **Vector Calculus** (NOT Quaternion)



$\hat{v}_t$: *Parallel to* $\hat{u}$

$\hat{v}_s$: *Perpendicular to* $\hat{u}$

$$\hat{v}' = R_\theta(\hat{v}) = R_\theta(\hat{v}_s) + R_\theta(\hat{v}_t)$$



Derive Vs

$$\therefore \hat{v}_t = (\hat{v}\cdot\hat{u})\hat{u}$$

$$\hat{v} = \hat{v}_t + \hat{v}_s$$

$$\therefore \hat{v}_s = \hat{v} - \hat{v}_t = \hat{v} - (\hat{v}\cdot\hat{u})\hat{u}$$

Derive w

$$\hat{w} = \hat{u}\times\hat{v}_s = \hat{u}\times\left(\hat{v}-\hat{v}_t\right) = \hat{u}\times\hat{v} - \hat{u}\times\hat{v}_t = \hat{u}\times\hat{v}$$

$$\because \hat{u} \parallel \hat{v}_t \rightarrow \hat{u}\times\hat{v}_t = 0,$$

Derive Rotation, $R_\theta(\hat{v}_s)$ in $\hat{v}_s, \hat{w}$ plane

$$R_\theta(\hat{v}_s) = \cos\theta\hat{v}_s + \sin\theta\hat{w}$$

$$R_\theta(\hat{v}_s) = \cos\theta\hat{v}_s + \sin\theta\hat{w}$$

$$R_\theta(\hat{v}_t) = \hat{v}_t = (\hat{v}\cdot\hat{u})\hat{u}$$

$$\hat{v}_s = \hat{v} - (\hat{v}\cdot\hat{u})\hat{u}$$

$$\hat{w} = \hat{u}\times\hat{v}$$

$$R_\theta(\hat{v}_s) = \cos\theta\left(\hat{v} - (\hat{v}\cdot\hat{u})\hat{u}\right) + \sin\theta\left(\hat{u}\times\hat{v}\right)$$

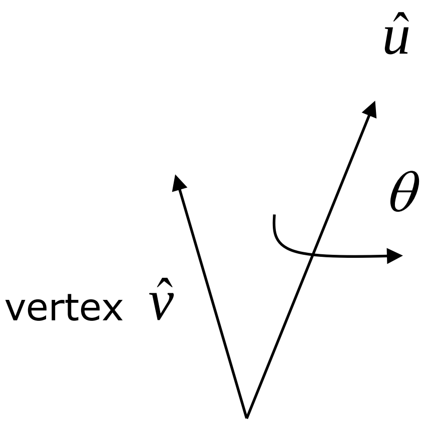
$$= \cos\theta\hat{v} - \cos\theta(\hat{v}\cdot\hat{u})\hat{u} + \sin\theta\left(\hat{u}\times\hat{v}\right)$$

$$\hat{v}' = R_\theta(\hat{v}) = R_\theta(\hat{v}_s) + R_\theta(\hat{v}_t)$$

$$= \cos\theta\hat{v} - \cos\theta(\hat{v}\cdot\hat{u})\hat{u} + \sin\theta\left(\hat{u}\times\hat{v}\right) + (\hat{v}\cdot\hat{u})\hat{u}$$

$$= \cos\theta\hat{v} + (1-\cos\theta)(\hat{v}\cdot\hat{u})\hat{u} + \sin\theta\left(\hat{u}\times\hat{v}\right)$$

Example of Rotation along an Arbitrary Axis by **Vector Calculus**



Arbitrary Rotation

$$\hat{v}' = \cos\theta \hat{v} + (1-\cos\theta)(\hat{v}\cdot\hat{u})\hat{u} + \sin\theta\left(\hat{u}\times\hat{v}\right)$$

- Think that x axis rotates along z axis with 90 degree.

$$\hat{x}' = 0\hat{x} + (1-0)(\hat{x}\cdot\hat{z})\hat{z} + 1\left(\hat{z}\times\hat{x}\right)$$
$$= 0 + 0\hat{z} + \hat{y} = \hat{y}$$

From Vector Calculus to Quaternion

$$if\ \ q=(s,\hat{u})=\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\hat{u}\right),\ \ |\hat{u}|=1$$

$$|q\cdot\bar{q}|=|s^2+\hat{u}\bullet\hat{u}|\ \ \rightarrow\ \ |q\cdot\bar{q}|=\cos^2\frac{\theta}{2}+\sin^2\frac{\theta}{2}\hat{u}\square\hat{u}=1$$

$$\textit{Lemma}:\ \hat{v}'=R_\theta(\hat{v})\ \square\ q\hat{v}q^{-1}$$

$$q^{-1}=\frac{\bar{q}}{\|q\|^2}=\bar{q}$$

$$\hat{v}'=q\hat{v}q^{-1}=q\hat{v}\bar{q}=\left(\cos\frac{\theta}{2}+\sin\frac{\theta}{2}\hat{u}\right)\hat{v}\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\hat{u}\right)$$

$$=\left(\cos\frac{\theta}{2}\hat{v}+\sin\frac{\theta}{2}\hat{u}\hat{v}\right)\left(\cos\frac{\theta}{2},-\sin\frac{\theta}{2}\hat{u}\right)$$

$$=\cos^2\frac{\theta}{2}\hat{v}+(\hat{u}\hat{v}-\hat{v}\hat{u})\cos\frac{\theta}{2}\sin\frac{\theta}{2}-\sin^2\frac{\theta}{2}\hat{u}\hat{v}\hat{u}$$

$$\hat{v}' = q\hat{v}q^{-1}$$

$$= \cos^2\frac{\theta}{2}\hat{v} + (\underline{\hat{u}\hat{v} - \hat{v}\hat{u}})\cos\frac{\theta}{2}\sin\frac{\theta}{2} - \sin^2\frac{\theta}{2}\underline{\hat{u}\hat{v}\hat{u}}$$

$$\hat{u}\hat{v} = (u_x i + u_y j + u_z k)(v_x i + v_y j + v_z k)$$

$$= -u_x v_x + u_x v_y k - u_x v_z j - u_y v_x k - u_y v_y + u_y v_z i + u_z v_x j - u_z v_y i - u_z v_z$$

$$\hat{v}\hat{u} = (v_x i + v_y j + v_z k)(u_x i + u_y j + u_z k)$$

$$= -v_x u_x + v_x u_y k - v_x u_z j - v_y u_x k - v_y u_y + v_y u_z i + v_z u_x j - v_z u_y i - v_z u_z$$

$$\hat{u}\hat{v} - \hat{v}\hat{u} = u_x v_y k - u_x v_z j - u_y v_x k + u_y v_z i + u_z v_x j - u_z v_y i - v_x u_y k + v_x u_z j + v_y u_x k - v_y u_z i - v_z u_x j + v_z u_y i$$

$$= (u_y v_z - u_z v_y - v_y u_z + v_z u_y)i + (-u_x v_z + u_z v_x + v_x u_z - v_z u_x)j + (u_x v_y - u_y v_x - v_x u_y + v_y u_x)k$$

$$= 2(\hat{u} \times \hat{v})$$

$$\therefore \hat{u}\hat{v} - \hat{v}\hat{u} = 2(\hat{u} \times \hat{v})$$

$$\therefore \hat{u}\hat{v}\hat{u} = \hat{v}(\hat{u} \cdot \hat{u}) - 2\hat{u}(\hat{u} \cdot \hat{v})$$

$$\hat{v}' = q\hat{v}q^{-1}$$

$$= \cos^2\frac{\theta}{2}\hat{v} + 2\cos\frac{\theta}{2}\sin\frac{\theta}{2}(\hat{u}\times\hat{v}) - \sin^2\frac{\theta}{2}\left(\hat{v}(\hat{u}\cdot\hat{u}) - 2\hat{u}(\hat{u}\cdot\hat{v})\right)$$

$$= \cos^2\frac{\theta}{2}\hat{v} + \sin\theta(\hat{u}\times\hat{v}) - \sin^2\frac{\theta}{2}\hat{v} + 2\sin^2\frac{\theta}{2}\hat{u}(\hat{u}\cdot\hat{v})$$

$$= \left(\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right)\hat{v} + \sin\theta(\hat{u}\times\hat{v}) + \left(1-\cos\theta\right)\hat{u}(\hat{u}\cdot\hat{v})$$

$$= \cos\theta\hat{v} + \sin\theta(\hat{u}\times\hat{v}) + \left(1-\cos\theta\right)\hat{u}(\hat{u}\cdot\hat{v})$$

$$= \cos\theta\hat{v} + \left(1-\cos\theta\right)\hat{u}(\hat{u}\cdot\hat{v}) + \sin\theta(\hat{u}\times\hat{v})$$

$$Lemma:\ \hat{v}' = R_\theta(\hat{v}) \equiv q\hat{v}q^{-1}$$

$$if\ \ q = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{u}\right),\quad |\hat{u}| = 1$$

$pp.13$, Vector Calculus

$$\hat{v}' = R_\theta(\hat{v}) = R_\theta(\hat{v}_s) + R_\theta(\hat{v}_t)$$

$$= \cos\theta\hat{v} - \cos\theta(\hat{v}\cdot\hat{u})\hat{u} + \sin\theta\left(\hat{u}\times\hat{v}\right) + (\hat{v}\cdot\hat{u})\hat{u}$$

$$= \cos\theta\hat{v} + (1-\cos\theta)(\hat{v}\cdot\hat{u})\hat{u} + \sin\theta\left(\hat{u}\times\hat{v}\right)$$

Quaternion is Too Complex??
Yes, It is.

- Let’s move to **Transform matrix from Quaternion**

Quaternion To Homogeneous Transform

$$\hat{v}' = R_\theta(\hat{v})$$

$$q = \left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\hat{u} \right), \quad |\hat{u}| = 1$$

$$q = \cos\frac{\theta}{2} + \sin\frac{\theta}{2}u_x i + \sin\frac{\theta}{2}u_y j + \sin\frac{\theta}{2}u_z k$$

$$= q_0 + q_1 i + q_2 j + q_3 k$$

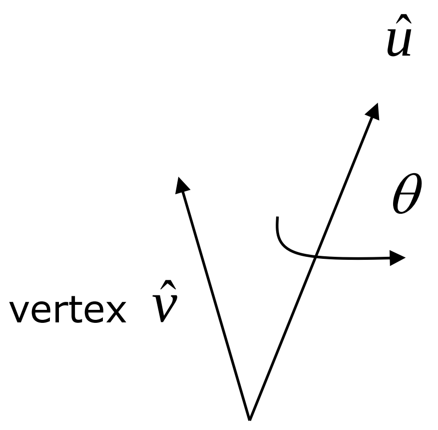
$$\therefore H = \begin{bmatrix} q_0^2 + q_1^2 - q_2^2 - q_3^2 & 2(q_1q_2 - q_0q_3) & 2(q_1q_3 + q_0q_2) & 0 \\ 2(q_1q_2 + q_0q_3) & q_0^2 - q_1^2 + q_2^2 - q_3^2 & 2(q_2q_3 - q_0q_1) & 0 \\ 2(q_1q_3 - q_0q_2) & 2(q_2q_3 + q_0q_1) & q_0^2 - q_1^2 - q_2^2 + q_3^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{v}' = H\hat{v} = R_\theta(\hat{v})$$

Three Results of Arbitrary Rotation

- Vector Calculus

$$\hat{v}' = \cos\theta\hat{v} + (1-\cos\theta)(\hat{v}\cdot\hat{u})\hat{u} + \sin\theta\left(\hat{u}\times\hat{v}\right)$$



- Quaternion Rotation

$$\hat{v}' = q\hat{v}q^{-1}$$

$$= \cos\theta\hat{v} + \left(1-\cos\theta\right)\hat{u}(\hat{u}\cdot\hat{v}) + \sin\theta(\hat{u}\times\hat{v})$$

- Homogenous Transform

$$H = \begin{bmatrix} q_0^2+q_1^2-q_2^2-q_3^2 & 2(q_1q_2-q_0q_3) & 2(q_1q_3+q_0q_2) & 0 \\ 2(q_1q_2+q_0q_3) & q_0^2-q_1^2+q_2^2-q_3^2 & 2(q_2q_3-q_0q_1) & 0 \\ 2(q_1q_3-q_0q_2) & 2(q_2q_3+q_0q_1) & q_0^2-q_1^2-q_2^2+q_3^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\hat{v}' = H\hat{v}$$

Quaternion→ Homogeneous Transform